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Reducible correlations in Dicke states

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Abstract
We apply a simple observation to show that the generalized Dicke states can be determined from their reduced subsystems. In this framework, it is sufficient to calculate the expression for only the diagonal elements of the reduced density matrices in terms of the state coefficients. We prove that the correlation in generalized Dicke states $\left|GD_N^{(\ell)}\right\rangle$ can be reduced to 2ℓ-partite level. Application to the quantum marginal problem is also discussed.

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1. Introduction

Entanglement is one of the most fascinating non-classical features of quantum theory which has been harnessed for various practical applications. Although bipartite entanglement is well understood, gaining insight into multipartite entanglement is still quite a challenge. There are various perspectives to study entanglement at the multiparty level such as its characterization by means of LOCC, its ability to reject local realism and hidden variable theories, etc. A particular interesting point of view is that of 'parts and whole'. This approach basically deals with the question: how much knowledge about the quantum system can be acquired from that of its subsystems? To be precise, it asks whether an unknown state can be determined *uniquely* if all its reduced density matrices (RDMs) are specified. In other words, this means whether higher order correlations are determined by lower order ones. It turns out that the most entangled states are the ones which cannot be determined from their RDMs.

The determination of a state from its RDMs implies that the correlation present in the state is reducible to lower order ones. In an interesting work [1] it was shown that except the GHZ class $(a|000\rangle + b|111\rangle)$, all 3-qubit pure states are determined by their 2-qubit RDMs. This was further generalized [2] to the N-qubit case to show that GHZ is the most entangled class of states. In these works, the knowledge of $(N-1)$-party RDMs was employed to characterize the N-party state. However, in the general scenario there may exist states which can be determined by less than $(N-1)$-party RDMs, i.e. a generic correlation can be reduced beyond the $(N-1)$-partite level. For example, we have recently shown [3] that the N-qubit W-class of states is determined by just their bipartite RDMs.

Though some partial progress has been made in this direction [4], there is no general technique to know which class of states can be determined by K-partite RDMs for $K < N-1$. Answering this question will lead to the classification of quantum states in terms of various kinds of reducible correlations that they can exhibit [1–3]. A natural way to solve the problem is to determine all the RDMs from the given state and from an arbitrary state (which is supposed to have the same set of RDMs) and then to compare the corresponding RDMs. But this is practically very difficult as we need to solve several second-degree equations involving complex numbers.

In this paper we provide some interesting examples of states which can be determined by their K-partite RDMs for $K < N-1$. In particular, we shall consider the Dicke states which are genuinely entangled and have been widely studied from both theoretical and experimental point of views [5]. The simplest Dicke state is the W-state which was studied at the qubit level recently [3]. In the present work, as a first step, we shall extend this result to arbitrary d-dimensional (i.e. N-qudit) W-state. Next, we shall focuss on the N-qubit Dicke states and study their reducible correlations. This result is further generalized to d-dimensions. Another interesting application of our technique that would be mentioned is the quantum marginal problem.

Our proof is based on the simple fact that the RDMs of a *pure* state can be constructed only from the expressions of the diagonal elements. This facilitates easy computation of the RDMs. In addition, if some of the diagonal entries are zero, then this constraints some diagonals of the arbitrary state to vanish, thereby reducing the number of unknowns. So first, let us rewrite some known observations and notational conventions to construct RDMs, in a slightly different way, for later convenience.

Observation 1. *To calculate the reduced density matrix from a generic pure state, it is sufficient to calculate the expression for diagonal elements in terms of the state coefficients. All off-diagonal elements will be obtained from these expressions.*

A density matrix being Hermitian can be identified by its upper-half elements $a_{ij} \forall i \leqslant j$. So we do not need to calculate the lower-half elements.

Using the 'lexicographically ordered' basis $\{|00\ldots0\rangle, |00\ldots1\rangle, \ldots, |00\ldots\overline{d-1}\rangle, \ldots, |\overline{d-1}\,\overline{d-1}\ldots\overline{d-1}\rangle\}$ of $\mathbb{C}^{d^{\otimes N}}$, an N-qudit pure state $|\psi\rangle_N^d$ (i.e. an N-partite pure quantum state where each of the parties has a d-level system) can be expressed as

$$|\psi\rangle_N^d = \sum_{i=0}^{d^N-1} c_i |D_N(i)\rangle, \qquad \sum_{i=0}^{d^N-1} |c_i|^2 = 1, \tag{1}$$

where $D_N(x) \equiv$ 'Representation of the decimal number x in an N-bit string in d-base number system'. To have a grip on the coefficient corresponding to a basis vector, we are using a d-base number system to represent the basis vector so that the suffix of its coefficient can be obtained by converting it into decimal number and vice versa. (Note that for $d \geqslant 11$, we need at least two bits (digit) to represent $d-1$ in a decimal number system. But we wish to restrict ourselves to using one bit to represent one level. So we are using the d-base number system to represent the bases. That is why a 'bar' is used over $d-1$ to indicate that it is of the d-base number system (and so it consists of one bit).)

Throughout the discussion, we will use the d-base number system to represent only the bases and decimal numbers elsewhere. However, when there is no ambiguity, we will write

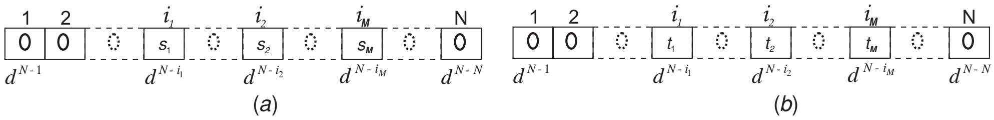

Figure 1. Least suffix (or the first term) in r_{ij} is $c_{k_0}\bar{c}_{l_0}$, where (*a*) $k_0 = \sum_{j=1}^{M} s_j \cdot d^{N-i_j}$ and (*b*) $l_0 = \sum_{j=1}^{M} t_j \cdot d^{N-i_j}$.

$|i\rangle$ instead of $|D_N(i)\rangle$—it should always be understood that the bases are in the d-base number system.

Now let us calculate the M-partite marginal (RDM) $\rho_\psi^{i_1 i_2 \ldots i_M} = Tr\big(|\psi\rangle_N^d\langle\psi|\big)$, where the trace is taken over the remaining $N{-}M$ parties. Clearly, it will be a matrix of order $d^M \times d^M$. So, retaining only the upper-half entries, we can write

$$\rho_\psi^{i_1 i_2 \ldots i_M} = \sum_{i=0}^{d^M-1} \sum_{j=i}^{d^M-1} r_{ij}|D_M(i)\rangle\langle D_M(j)|. \tag{2}$$

Since the RDM is obtained by tracing over $N{-}M$ parties (the space of these parties has dimension d^{N-M}), each r_{ij} will be a sum of d^{N-M} number of terms each of which is of the form $c_k\bar{c}_l$. Thus, $r_{ij} = \sum_{p=0}^{d^{N-M}-1} c_{k_p}\bar{c}_{l_p}$. To get the expression of r_{ij} (i.e. to see explicitly which c_k's and c_l's will appear in the sum), let us fix one i and one j. Let $D_M(i) = s_1 s_2 \ldots s_M$ and $D_M(j) = t_1 t_2 \ldots t_M$. In an N-bit string, let us now put the s_j's at i_jth places respectively. Then the suffixes k's will be obtained by converting the N-bit d-base numbers, obtained by filling all the remaining $N{-}M$ places of the above string arbitrarily with $0, 1, 2, \ldots, \overline{d-1}$, into decimal numbers. For an illustration, let k_0 be the decimal number obtained by converting the N-bit d-base number having s_j fixed at the i_jth place ($\forall j = 1(1)M$) and zero at all the remaining $N{-}M$ places (see figure 1 for illustration). Then $k_0 = \sum_{j=1}^{M} s_j \cdot d^{N-i_j}$. Similarly, let $l_0 = \sum_{j=1}^{M} t_j \cdot d^{N-i_j}$. Then the first term (ordering in suffixes appear) of the sum in the expression of r_{ij} will be $c_{k_0} \cdot \bar{c}_{l_0}$. In a similar way other $c_{k_p} \cdot \bar{c}_{l_p} (p = 0(1)(d^{N-M} - 1))$ terms can be calculated. For the last term, which is the term with highest suffix, we have $k_{d^{N-M}-1} = k_0 + (d-1)\sum_{j=1;\, j\neq i_1,i_2,\ldots,i_M}^{N} d^{N-j}$. Note that $r_{ii} = \sum_{p=0}^{d^{N-M}} |c_{k_p}|^2$ and $r_{jj} = \sum_{p=0}^{d^{N-M}} |c_{l_p}|^2$. Since $r_{ij} = \sum_{p=0}^{d^{N-M}} c_{k_p} \cdot \bar{c}_{l_p}$, it follows that each off-diagonal element r_{ij} can be obtained by summing the products of corresponding complex numbers appearing in the expression of r_{ii} and conjugate of the complex numbers appearing in the expression of r_{jj}. Hence, it is sufficient to calculate the expression for only the diagonal elements.

Remark 1. If we start with a mixed state $[r_{ij}]_{j\geqslant i=0}^{d^N-1}$ and we wish to calculate an RDM $[R_{ij}]_{j\geqslant i=0}^{d^M-1}$, following the same procedure, it can be shown that if $R_{ii} = \sum_{s=0}^{d^M-1} r_{(k_s)(k_s)}$ and $R_{jj} = \sum_{s=0}^{d^M-1} r_{(l_s)(l_s)}$, then $R_{ij} = \sum_{s=0}^{d^M-1} r_{(k_s)(l_s)}$. So $R_{ii} = 0$ would imply $r_{(k_s)(j)} = 0\,\forall s, j$! Thus, it is always helpful to calculate first the expression of the diagonal elements (and compare them).

We shall now apply the above observation to arrive at our main results. As a natural extension of the work on N-qubit W-state [3], first we shall consider the generalized d-dimensional W-state. This would also serve as a good demonstration of the technique and be useful in understanding the proof for the generalized Dicke states.

2. *N*-qudit generalized *W*-state

The *N*-qudit generalized *W*-state is defined as [6]

$$|W\rangle_N^d = \sum_{i=1}^{d-1}(a_{1i}|i0\ldots00\rangle + \cdots + a_{ni}|00\ldots0i\rangle). \tag{3}$$

However, we will write this state in our notation as

$$|W\rangle_N^d = \sum_{i=0}^{N-1}\sum_{j=1}^{d-1} w_{jd^i}|D_N(jd^i)\rangle, \qquad \sum_{i=0}^{N-1}\sum_{j=1}^{d-1}|w_{jd^i}|^2 = 1. \tag{4}$$

Theorem 1. *N-qudit generalized W-states are determined by their bipartite marginals.*

We shall prove this by showing that there does not exist any other *N*-qudit density matrix having the same bipartite marginals except

$$\begin{aligned}|W\rangle_N^d\langle W| &= \sum_{i=0}^{N-1}\sum_{j=1}^{d-1}\sum_{k=j}^{d-1} w_{jd^i}\bar{w}_{kd^i}|D_N(jd^i)\rangle\langle D_N(kd^i)| \\ &\quad + \sum_{i=0}^{N-1}\sum_{j=1}^{d-1}\sum_{l=i+1}^{N-1}\sum_{m=1}^{d-1} w_{jd^i}\bar{w}_{jd^l}|D_N(jd^i)\rangle\langle D_N(md^l)|. \end{aligned} \tag{5}$$

(Note that though the above expression looks rather cumbersome, the matrix form can be easily visualized as there are non-zero elements only at (jd^i+1, kd^l+1) positions where $j,k = 1(1)(d-1); i,l = 0(1)(N-1)$ and $jd^i \leqslant kd^l$, since we are considering only the upper-half elements. These elements are the coefficients of $|D_N(jd^i)\rangle\langle D_N(kd^l)|$ and are given by $w_{jd^i}\bar{w}_{kd^l}$.)

Proof.

(1) Each bipartite marginal ρ_W^{JK} will be a matrix of order $d^2\times d^2$ where $J\in\{1,2,\ldots,N-1\}$ and $K\in\{2,3,\ldots,N\}$. As discussed earlier, to determine ρ_W^{JK}, we need to find the expressions of the d^2 diagonal elements of ρ_W^{JK}, i.e. the coefficients of $|ij\rangle\langle ij|\forall i,j = 0(1)(d-1)$. (Note that, while in the bases, i, j should be understood as d-base numbers).

Now each basis state in (4) has exactly one non-zero entry (rather 'bit') with value 1 to $d-1$. So there will be no basis term $|ij\rangle\langle ij|$ of ρ_W^{JK} having both i, j as non-zero numbers. Therefore, the coefficient of $|ij\rangle\langle ij|$ in ρ_W^{JK} should be zero $\forall i, j = 1(1)(d-1)$. Hence, we need to consider only the coefficients of $|0i\rangle\langle 0i|$ and $|i0\rangle\langle i0|$ $\forall i = 0(1)(d-1)$.

Again, for $i\neq 0$, there is exactly one basis state in (4) containing 'i' at the Jth place (from left to right) having the coefficient $w_{id^{N-J}}$. Therefore, the coefficient of $|i0\rangle\langle i0|$ is $|w_{id^{N-J}}|^2$. Similarly, the coefficient of $|0i\rangle\langle 0i|$ in ρ_W^{JK} is $|w_{id^{N-K}}|^2$ $\forall i = 1(1)(d-1)$. From normalization ($Tr(\rho_W^{JK}) = 1$), the coefficient of $|00\rangle\langle 00|$ in ρ^{JK} is obtained as

$$1 - \sum_{i=1}^{d-1}(|w_{id^{N-J}}|^2 + |w_{id^{N-K}}|^2).$$

We know that the non-diagonal terms (the coefficients of $|0i\rangle\langle 0j|, |0i\rangle\langle j0|$ and $|i0\rangle\langle j0|$; $i\neq j$) will be determined from the above expressions of the diagonal terms.

Thus,

$$\rho_W^{JK} = \left(1 - \sum_{i=1}^{d-1}(|w_{id^{N-J}}|^2 + |w_{id^{N-K}}|^2)\right)|00\rangle\langle 00| + \sum_{i=1}^{d-1}\sum_{j=i}^{d-1} w_{id^{N-K}}\bar{w}_{jd^{N-K}}|0i\rangle\langle 0j|$$
$$+ \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} w_{id^{N-K}}\bar{w}_{jd^{N-J}}|0i\rangle\langle j0| + \sum_{i=1}^{d-1}\sum_{j=i}^{d-1} w_{id^{N-J}}\bar{w}_{jd^{N-J}}|i0\rangle\langle j0|. \qquad (6)$$

(2) Now let us suppose that there exists an N-qudit density matrix (possibly mixed, hence the subscript M)

$$\rho_M^{12\ldots N} = \sum_{i=0}^{d^N-1}\sum_{j=i}^{d^N-1} r_{ij}|D_N(i)\rangle\langle D_N(j)| \qquad (7)$$

which has the same bipartite marginals as $|W\rangle_N^d$. Here $r_{ii} \geqslant 0\,\forall i = 0(1)(d^N-1)$, since the diagonal elements of a positive semi-definite (PSD) matrix are non-negative. (If possible let in a PSD matrix A, a diagonal element $d_i < 0$. Then taking $|\psi\rangle = [0, 0, \ldots, 0, 1, 0, \ldots, 0]^T$, we have $\langle\psi|A|\psi\rangle = d_{ii} < 0$, a contradiction that A is PSD.)

We first wish to calculate the diagonal elements of the bipartite marginal ρ_M^{JK} of $\rho_M^{12\ldots N}$. Each diagonal element of ρ_M^{JK} is the sum of the d^{N-2} number of diagonal elements r_{ss} of $\rho_M^{12\ldots N}$. Out of the d^2 number of diagonal elements (coefficients of $|ij\rangle\langle ij|\forall i, j = 0(1)(d-1)$) of ρ_M^{JK}, let us first calculate the coefficient of $|ij\rangle\langle ij|\forall i, j = 1(1)(d-1)$. To see explicitly which r_{ss}'s will appear in the sum, we observe that the suffixes s will vary over the decimal numbers obtained by converting the N-bit d-base numbers having i fixed at Jth and j fixed at Kth places and arbitrarily $0, 1, \ldots, \overline{d-1}$ at the remaining $(N-2)$ places. Hence, the terms r_{ss}'s for the suffixes $s = 0$ and $s = k \cdot d^{l-1}, k = 1, 2, \ldots, (d-1); l = 1, 2, \ldots, N$, will not appear in the expression (sum) of coefficient of $|ij\rangle\langle ij|$ in ρ_M^{JK} for any J, K as for these s, $D_N(s)$ can have at most one non-zero entry (but we need at least two).

(3a) As can be seen from equation (6), there is no term $|ij\rangle\langle ij|$ for $ij \neq 0$ in ρ_W^{JK}. Therefore, the coefficient of $|ij\rangle\langle ij|$ for $ij \neq 0$ in ρ_M^{JK} should vanish. Since these coefficients are sum of non-negative r_{ss}'s, each r_{ss} appearing there should individually be zero. Therefore, from step (2), the only non-zero diagonal elements of $\rho_M^{12\ldots N}$ are r_{ii} for $i = 0$ and $i = j \cdot d^{k-1}\forall j = 1(1)(d-1), k = 1(1)N$.

(3b) Next comparing the coefficient of $|0i\rangle\langle 0i|$ from ρ_W^{JK} and ρ_M^{JK}, we get $r_{(id^{N-K})(id^{N-K})} = |w_{id^{N-K}}|^2$ for all $i = 1(1)(d-1)$. Similarly, comparing the coefficient of $|i0\rangle\langle i0|$, we get $r_{(id^{N-J})(id^{N-J})} = |w_{id^{N-J}}|^2$ for all $i = 1(1)(d-1)$. Since these results hold for all possible (parties) J and K, we can write them in combined form as $r_{(jd^i)(jd^i)} = |w_{jd^i}|^2\forall j = 1(1)(d-1), i = 0(1)(N-1)$.

(3c) Finally, from the normalization condition $\sum_{i=0}^{d^N-1} r_{ii} = 1 = \sum_{i=0}^{N-1}\sum_{j=1}^{d-1}|w_{jd^i}|^2$, we get $r_{00} = 0$. Thus, collecting the results of steps 3a and 3b it follows that

$$r_{(jd^i)(jd^i)} = |w_{jd^i}|^2, \qquad \forall j = 1(1)(d-1), \qquad i = 0(1)(N-1), \qquad (8)$$

and all other r_{ii} in $\rho_M^{12\ldots N}$ are zero.

(4) Now we will use the fact that if a diagonal element of a PSD matrix is zero, then all elements in the row and column containing that element should be zero [7]. Hence, from the result of step 3c it follows that $\rho_M^{12\ldots N}$ has non-zero elements only at (jd^i+1, kd^l+1) positions where $j, k = 1(1)(d-1); i, l = 0(1)(N-1)$ and $jd^i \leqslant kd^l$. These elements are the coefficients of $|D_N(jd^i)\rangle\langle D_N(kd^l)|$ and are given by $r_{(jd^i)(kd^l)}$. Therefore, $\rho_M^{12\ldots N}$

has the same form as $|W\rangle_N^d\langle W|$ given in (5). Moreover, from (8), they have the same diagonal elements.

(5) The non-diagonal elements of $\rho_M^{12\ldots N}$ are at (jd^i+1, kd^l+1) places with $jd^i < kd^l$. Now, $jd^i < kd^l$ may happen in two ways: either $i < l$ or $j < k$ (when $i = l$). For $i < l$, the non-diagonal element at (jd^i+1, kd^l+1) is found to be $w_{jd^i}\bar{w}_{kd^l}$ by comparing the coefficients of $|0j\rangle\langle k0|$ from $\rho_M^{(N-l)(N-i)}$ and $\rho_W^{(N-l)(N-i)}$. For $i = l$ (and hence $j < k$), the same can be achieved by comparing the coefficients of $|0j\rangle\langle 0k|$ from $\rho_M^{J(N-i)}$ and $\rho_W^{J(N-i)}$ with any $J \neq i$. Thus, $r_{(jd^i)(kd^l)} = w_{jd^i}\bar{w}_{kd^l}$ and hence $\rho_M^{12\ldots N} = |W\rangle_N^d\langle W|$. □

3. N-qubit generalized Dicke states

The generalized Dicke states are defined by

$$\left|GD_N^{(\ell)}\right\rangle = \sum_i a_i|i\rangle \tag{9}$$

where $i = |i_1 i_2 \ldots i_N\rangle$ and the sum varies over all permutations of ℓ number of 1 and $N-\ell$ number of 0. When all coefficients are equal, they are known as Dicke states which have many interesting properties such as permutational invariance, robustness against decoherence, measurement and particle loss. Some important applications of Dicke states include telecloning, quantum secret sharing, open-destination teleportation and quantum games [8]. In particular, implementation and various interesting applications of W-states and their connection with Dicke states have been studied in [9]. Thus, Dicke states serve as a good test bed for exploring multiparty correlations.

Let us first write the above state in (9) as (to have a grip on the coefficients)

$$\left|GD_N^{(\ell)}\right\rangle = \sum_{i_1>i_2>\ldots>i_\ell=0}^{N-1} a_{2^{i_1}+2^{i_2}+\cdots+2^{i_\ell}}\left|B_N\left(2^{i_1}+2^{i_2}+\cdots+2^{i_\ell}\right)\right\rangle \tag{10}$$

where $B_N(x)$ is the binary representation of the decimal number x in an N-bit string and a_i's are arbitrary non-zero complex numbers satisfying the normalization condition.

Retaining only the upper-half elements, we can write

$$\left|GD_N^{(\ell)}\right\rangle\!\left\langle GD_N^{(\ell)}\right| = \sum_{i\leqslant j} g_{ij}|B_N(i)\rangle\langle B_N(j)| \tag{11}$$

where $g_{ij} = a_i\bar{a}_j$ and i, j vary over the decimal numbers obtained by converting the N-bit binary numbers having ℓ number of 1 (and $N-\ell$ numbers of 0). In matrix form $\left|GD_N^{(\ell)}\right\rangle\!\left\langle GD_N^{(\ell)}\right|$ will have non-zero entries (g_{ij}) only at $(i+1, j+1)$ positions.

Since $1 \leqslant \ell \leqslant N-1$, considering the entanglement, it is sufficient to take $\ell \leqslant \left\lfloor \frac{N}{2} \right\rfloor =$ *integral part* of $\frac{N}{2}$, as the states corresponding to other ℓ's are LU-equivalent to these states. Any property of these latter states will follow from the corresponding former states obtained by changing 0 and 1 throughout the bases. For example, the two classes $\left|GD_N^{(N-2)}\right\rangle$ and $\left|GD_N^{(2)}\right\rangle$ have the same property. We shall now prove an interesting property of these states.

Theorem 2. *For $1 \leqslant \ell < \left\lfloor \frac{N}{2} \right\rfloor$, the generalized Dicke state $\left|GD_N^{(\ell)}\right\rangle$ is uniquely determined by its 2ℓ-partite marginals.*

Note that we have excluded the case $\ell = \left\lfloor \frac{N}{2} \right\rfloor$. The reason for this exclusion will be described later. We will prove this theorem in two parts—firstly we shall show that if any density matrix has the same $(\ell+1)$-partite marginals as those of $\left|GD_N^{(\ell)}\right\rangle$, then it must share the same diagonal elements with $\left|GD_N^{(\ell)}\right\rangle\!\left\langle GD_N^{(\ell)}\right|$. But there will be some (off-diagonal) elements in a general density matrix which will never appear in any $(\ell+1)$-partite marginal. So, to include these elements, we have to consider RDMs of more parties. In the second part we will show that it is sufficient to consider the 2ℓ-partite marginals to prove the uniqueness (i.e. the two matrices share the same off-diagonals).

Proof.

(1) If possible, let there exists an N-qubit density matrix (possibly mixed)

$$\rho_M^{12\ldots N} = \sum_{i=0}^{2^N-1}\sum_{j=i}^{2^N-1} r_{ij}|B_N(i)\rangle\langle B_N(j)| \tag{12}$$

having the same $(\ell+1)$-partite marginals as those of $\left|GD_N^{(\ell)}\right\rangle$. We shall prove that $\rho_M^{12\ldots N}$ and $\left|GD_N^{(\ell)}\right\rangle\!\left\langle GD_N^{(\ell)}\right|$ share the same diagonals.

(2) We will first consider the diagonal elements of RDMs. Since there is exactly ℓ number of non-zero entry (each is 1) in every basis term of $\left|GD_N^{(\ell)}\right\rangle$, the coefficient of

$$|i_1 i_2 \ldots i_{\ell+1}\rangle\langle i_1 i_2 \ldots i_{\ell+1}|$$

in any $(\ell+1)$-partite marginal should be zero in which every i_k is 1. This constraints the form of $\rho_M^{12\ldots N}$ in equation (12) to have some coefficients (r_{ij}) vanishing. In (12), only those r_{ij} will be non-zero for which both of $B_N(i)$ and $B_N(j)$ have at most ℓ number of 1. We shall now show that only those r_{ii} in (12) are non-zero for which $B_N(i)$ has exactly ℓ number of 1.

(2a) Let us consider the coefficient of $|i_1 i_2 \ldots i_{\ell+1}\rangle\langle i_1 i_2 \ldots i_{\ell+1}|$ where ℓ number of i_j's is 1 and only one is zero, in the RDM of some parties $J_1, J_2, \ldots, J_{\ell+1}$. There is exactly one basis term in $\left|GD_N^{(\ell)}\right\rangle$ having i_k at the J_kth place, with the coefficient a_i where $i = 2^{N-J_{\ell+1}} + \cdots + 2^{N-J_1}$. So, when the RDM is calculated from $\left|GD_N^{(\ell)}\right\rangle$, the coefficient is $g_{ii} = |a_i|^2$. Again, there is exactly one non-zero r_{ii} in (12) such that $B_N(i)$ has i_k at the J_kth place (since $B_N(i)$ can have at most ℓ number of 1, in order for $r_{ij} \neq 0$). Therefore, comparing the coefficients (of this term), $r_{ii} = g_{ii}$. Considering all permutations of this term and all possible set of $(\ell+1)$ number of parties, it follows that $r_{ii} = d_{ii}$, for all decimal i so that $B_N(i)$ has ℓ number of 1.

(2b) Now we will show that all other r_{ii}, corresponding to which $B_N(i)$ has less than ℓ number of 1, should be 0. First consider those r_{ii}'s corresponding to which $B_N(i)$ has $(\ell-1)$ number of 1. Then comparing the coefficients of $|i_1 i_2 \ldots i_{\ell+1}\rangle\langle i_1 i_2 \ldots i_{\ell+1}|$ where $(\ell-1)$ number of i_j's are 1, from the RDMs (considering all possible set of parties and using the result of step (a) above), we get $r_{ii} = 0$. Similarly, all other r_{ii}'s, corresponding to which $|B_N(i)\rangle$ has less than ℓ number of 1, should be zero. Finally from normalization $\left(\sum r_{ii} = \sum g_{ii} = 1\right)$, it follows that $r_{00} = 0$.

Thus, collecting the results of (a) and (b) it follows that $\rho_M^{12\ldots N}$ in (12) reduces to the same form as $\left|GD_N^{(\ell)}\right\rangle\!\left\langle GD_N^{(\ell)}\right|$ in (11) and they have the same diagonal elements $r_{ii} = g_{ii}$. The only remaining task to prove the uniqueness is to show that they have the same non-diagonal elements too.

(3) Consider a non-diagonal element r_{ij} with $i = |\ldots i_1 \ldots i_\ell \ldots\rangle$ and $j = |\ldots j_1 \ldots j_\ell \ldots\rangle$ (each of i_k and j_k is 1). Since $\ell < \left\lfloor \frac{N}{2} \right\rfloor$, there will be some terms $|i\rangle\langle j|$ in the density matrix, the coefficients (r_{ij} or $a_i\bar{a}_j$) of which will never occur in any $(\ell+1)$-partite marginal. For example, the coefficient of $|000\ldots011\rangle\langle 110\ldots 0|$, or $|010\ldots01\rangle\langle 100\ldots010|$ will never appear in any tripartite marginal. Generically, those r_{ij}'s $(i < j)$ for which the Hamming distance between $B_N(i)$ and $B_N(j)$ is greater than $\ell+1$ will never occur in any

$(\ell+1)$-partite marginal; because partial tracing over the remaining parties will yield 0. Thus, the elements r_{ij}'s with $j = \bar{i}$=*complement* of $i \equiv 2^N - 1 - i$ (these are the elements on the secondary diagonal of the density matrix) will never occur.

Since these r_{ij}'s never occur in any $(\ell+1)$-partite marginal, these are unconstrained elements (i.e. these can take any values and need not be $a_i\bar{a}_j$, which is required for the uniqueness of the two density matrices). So, there exists an infinite number of $2^N \times 2^N$ Hermitian, unit-trace matrices sharing the same diagonals and $(\ell+1)$-partite marginals with $|GD_N^{(\ell)}\rangle\langle GD_N^{(\ell)}|$. However, all such matrices may not be valid density matrices because of the semi-positivity restriction ($\rho \geqslant 0$). So, for some particular choices of the coefficients a_i's, there may (or may not) exist a valid density matrix other than $|GD_N^{(\ell)}\rangle\langle GD_N^{(\ell)}|$. Therefore, there is an ambiguity about the general case: what is the minimum number of parties whose RDM's can generically determine the $|GD_N^{(\ell)}\rangle$ state uniquely?

(4) To answer this question, we observe that the possible maximum Hamming distance between $B_N(i)$ and $B_N(j)$ is 2ℓ. Therefore, if we consider the 2ℓ-partite marginals, each r_{ij} will appear in some RDMs and hence should be constrained to satisfy some relation with a_i's. We shall now show that considering 2ℓ-partite marginals indeed yield $r_{ij} = a_i\bar{a}_j$.

To prove it, consider a non-diagonal element r_{ij} with $i = |i_1 i_2 \ldots i_N\rangle$ and $j = |j_1 j_2 \ldots j_N\rangle$. Let the ℓ 1's in i be at I_kth places (counting from left to right) and those in j are at J_kth places. If the two sets $\{I_k\}$ and $\{J_k\}$ are different (i.e. $\{I_k\} \cap \{J_k\} = \Phi$), then we get a set of 2ℓ number of parties $\{I_k, J_k\}$ and we can arrange all I_k and J_k's (since $I_k, J_k \in \{1, 2, \ldots, N\}$) in increasing order. Let us call them $\{P_k\}$ (i.e. $P_1 < P_2 < \ldots < P_{2\ell}$). If $\{I_k\} \cap \{J_k\} \neq \Phi$, we can add any number(s) from $\{1, 2, \ldots, N\}$ to the set $\{P_k\}$ (maintaining the order) so that it contains 2ℓ number of elements. Let s_k be the P_kth bit (from left to right) in $B_N(i)$ and those in $B_N(j)$ be t_k. Then comparing the coefficient of $|s_1 s_2 \ldots s_{2\ell}\rangle\langle t_1 t_2 \ldots t_{2\ell}|$ from the RDMs $\rho^{P_1 P_2 \ldots P_{2\ell}}$, it follows that $r_{ij} = a_i\bar{a}_j$ and hence the proof. □

Remark 2. It is worth mentioning that theorem 2 can be viewed as a sufficient condition. It states that it is sufficient to consider the 2ℓ-partite marginals to determine $|GD_N^{(\ell)}\rangle$. However, it may happen (e.g. for some specific state in this class) that the state $|GD_N^{(\ell)}\rangle$ can be determined from fewer than 2ℓ-partite marginals. In this sense, we do not know whether this is an optimal bound. We have used the 2ℓ-partite marginals to drive out the possibility of the presence of another density matrix having different off-diagonals but sharing the same diagonals. The off-diagonals r_{ij} are arbitrary but are constrained to satisfy the requirement that the resulting matrix should be PSD. This automatically puts some restrictions on the off-diagonals e.g. $|r_{ij}| \leqslant \sqrt{r_{ii} r_{jj}}$. There is a possibility of reducing the number of parties using some further properties of density (PSD) matrices (or using some different techniques). In the present technique, 2ℓ-partite marginals are sufficient. A limitation of the present technique is that if the maximum Hamming distance (between the bases) is N, then it gives the trivial result. In the case $\ell = \lfloor \frac{N}{2} \rfloor$, for odd N, the technique supports the result of [2] and for even N, it gives no useful result. That is why we have excluded this case in theorem 2.

Another interesting issue is the number of RDMs needed to identify a state. For example, it has been shown by Diosi [4] that among pure states, only two (out of three) bipartite marginals are sufficient to determine a generic 3-qubit pure state (GHZ and its LU equivalents are the only exception). If we restrict ourselves only to pure states, then the number of

RDMs can be considerably reduced. The result is stated more precisely in the following theorem.

Theorem 3. *Among arbitrary pure states, the generalized Dicke state* $\left|GD_N^{(\ell)}\right\rangle$ *is uniquely determined by its* $(\ell+1)$*-partite marginals. Moreover, only* ${}^{N-1}C_\ell$ *(out of* ${}^{N}C_{\ell+1}$*) number of them having one party common to all are sufficient.*

Proof. Let us take the first party as the common one and consider the following RDMs $\rho^{1i_2i_3\ldots i_{\ell+1}}$. The proof for diagonal part has already been established in the first part of theorem 2. The proof for the non-diagonal part follows by comparing the coefficients of $|B_{\ell+1}(i)\rangle\langle B_{\ell+1}(j)|$, where $|B_{\ell+1}(i)\rangle$ and $|B_{\ell+1}(j)\rangle$ have exactly ℓ number of 1s. □

Remark 3. We wish to mention here that as we are considering the most general class of Dicke states, it is not possible to determine the states from fewer than $(\ell+1)$-partite marginals. It may happen that for some specific choices of the coefficients, $\left|GD_N^{(\ell)}\right\rangle$ is uniquely determined from fewer than $(\ell+1)$-partite marginals, but in general, not all states can be determined. For example, the following two states:

$$\left|GD_4^{(2)}\right\rangle = r_3\,\mathrm{e}^{\mathrm{i}\theta_3}(|3\rangle + |12\rangle) + r_5\,\mathrm{e}^{\mathrm{i}\theta_5}(|5\rangle + |10\rangle) + r_6\,\mathrm{e}^{\mathrm{i}\theta_6}(|6\rangle + |9\rangle)$$

$$\text{and}\quad \left|GD_4^{(2)'}\right\rangle = r_3\,\mathrm{e}^{-\mathrm{i}\theta_3}(|3\rangle + |12\rangle) + r_5\,\mathrm{e}^{-\mathrm{i}\theta_5}(|5\rangle + |10\rangle) + r_6\,\mathrm{e}^{-\mathrm{i}\theta_6}(|6\rangle + |9\rangle)$$

are not determinable since they share the same bipartite marginals [r_i, θ_i are real and the base $|x\rangle$ should be read as $|B_4(x)\rangle$].

4. Generalization to *d*-dimension

The generalized d-dimensional Dicke states are defined by

$$|D_N(k_0, k_1, \ldots, k_{d-1})\rangle = \sum_i c_i |i\rangle \tag{13}$$

where

$$i = |\underbrace{0\ldots0}_{k_0}\underbrace{1\ldots1}_{k_1}\ldots\underbrace{\overline{d-1}\ldots\overline{d-1}}_{k_{d-1}}\rangle \tag{14}$$

and the index i varies over all possible different permutations of k_0 number of 0, k_1 number of $1, \ldots, k_{d-1}$ number of $\overline{d-1}$; $k_0 + k_1 + \cdots + k_{d-1} = N$. These states are genuinely entangled. Using the same technique as in the proof of theorem 2, we can prove the following result about the reducible correlations in these states.

Theorem 4. *If* $K(\equiv 2m < N)$ *be the maximum Hamming distance between the bases (14), the state given by (13) is uniquely determined by its K-partite RDMs.*

As an example, any state of the class $|D_{2009}(2004, 2, 3)\rangle$ is determined by its 10-partite RDMs.

5. Quantum marginal problem

The basic issue concerning the quantum marginal problem (QMP) is the following: does there exist a joint quantum state consistent with a given set of RDMs? It is known that a general solution to the QMP would provide a solution to the *N-representability problem* in quantum chemistry, e.g. to calculate the binding energies of complex molecules [10]. A particular class of QMP is 'symmetric extension', which has a direct application in quantum key sharing, quantum cryptography, etc [11]. Although plenty of literature is available [12], there is no general method to get the exact solutions. One needs to calculate the marginals from an arbitrary state (which is the expected joint state) and then compare with the given ones. For a large number of marginals, the problem becomes very difficult as we need to solve several complex equations. However, if there is some symmetry in the RDMs (e.g. for W-states and Dicke states, all RDMs have similar form with many vanishing elements), the technique presented in our work can be applied to find a solution. As an interesting example, it was mentioned in [1] that for the set of RDMs $\{\rho^{AB}, \rho^{BC}, \rho^{AC}\}$ where (in computational basis)

$$\rho^{AB} = \rho^{BC} = \rho^{AC} = |\psi^-\rangle\langle\psi^-| \tag{15}$$

(with $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$) there exists no consistent 3-qubit state. We can prove this easily using our technique. If possible, let

$$\rho^{ABC} = \sum_{i=0}^{7}\sum_{j=i}^{7} r_{ij}|B_3(i)\rangle\langle B_3(j)| \tag{16}$$

where $B_3(x)$ =‘Binary representation of x in a 3-bit string’ has the given marginals. Then comparing the first and last diagonal elements (i.e. the coefficients of $|00\rangle\langle 00|$ and $|11\rangle\langle 11|$) of the RDMs, we get $r_{ii} = 0\,\forall i = 0(1)7$, an impossibility!

6. Conclusions

Though the general framework is still far way, through this work we have made considerable progress towards understanding the nature of reducible correlations. It has been shown that the correlations in some classes of multipartite states can be reduced to lower order ones. This provides some insight into the characterization of multiparty entanglement, such as the determination of generalized W-state from its bipartite RDMs proves that the entanglement therein is necessarily of bipartite nature. The large class of generalized Dicke states $\left|GD_N^{(\ell)}\right\rangle$ has shown to be determined by their 2ℓ-partite marginals, where $1 \leqslant \ell < \left\lfloor \frac{N}{2} \right\rfloor$. Thus, these states have information at the most at 2ℓ-partite level and it cannot be reduced beyond $(\ell + 1)$-partite level.

In general, the entangled states which are determined by their K-party RDMs can be used as resources for performing information related tasks, specially if some of the parties do not cooperate. In such situations, it is not necessary that each party cooperates with all others; cooperation with only $K - 1$ parties is required. The K-partite residual entanglement would serve the purpose. For example, because of the bipartite nature of entanglement, the N-qubit W-state is very robust against the loss of $(N - 2)$ parties.

Recently, it has been shown that $(N - 1)$-qudit RDMs uniquely determine the Groverian measure of entanglement of the N-qudit pure state [13]. So it is likely that for the pure states, which are determined by their K-partite RDMs, the entanglement measure may be characterized by these RDMs. However, this requires further investigation.

Finally, we have shown by an example that our approach can be applied to quantum marginal problem, at least for simple (low-dimensional) cases.

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